

# New characterizations of minimum spanning trees and of saliency maps based on quasi-flat zones<sup>\*</sup>

Jean Cousty<sup>1</sup>, Laurent Najman<sup>1</sup>, Yukiko Kenmochi<sup>1</sup>, Silvio Guimarães<sup>1,2</sup>

<sup>1</sup> Université Paris-Est, LIGM, A3SI, ESIEE Paris, CNRS  
 {j.cousty, l.najman, y.kenmochi, s.guimaraes}@esiee.fr,  
<sup>2</sup> PUC Minas - ICEI - DCC - VIPLAB

**Abstract.** We study three representations of hierarchies of partitions: dendrograms (direct representations), saliency maps, and minimum spanning trees. We provide a new bijection between saliency maps and hierarchies based on quasi-flat zones as used in image processing and characterize saliency maps and minimum spanning trees as solutions to constrained minimization problems where the constraint is quasi-flat zones preservation. In practice, these results form a toolkit for new hierarchical methods where one can choose the most convenient representation. They also invite us to process non-image data with morphological hierarchies.

**Keywords:** Hierarchy, saliency map, minimum spanning tree

## 1 Introduction

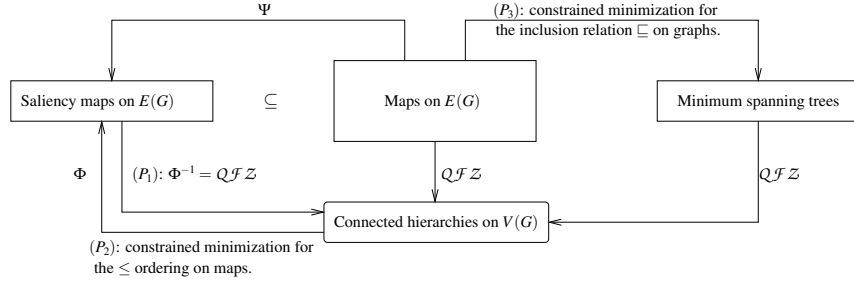
Many image segmentation methods look for a partition of the set of image pixels such that each region of the partition corresponds to an object of interest in the image. Hierarchical segmentation methods, instead of providing a unique partition, produce a sequence of nested partitions at different scales, enabling to describe an object of interest as a grouping of several objects of interest that appear at lower scales (see references in [14]). This article deals with a theory of hierarchical segmentation as used in image processing. More precisely, we investigate different representations of a hierarchy: by a dendrogram (direct set representation), by a saliency map (a characteristic function), and by a minimum spanning tree (a reduced domain of definition). Our contributions are threefold:

1. a new bijection theorem between hierarchies and saliency maps (Th. 1) that relies on the quasi-flat zones hierarchies and that is simpler and more general than previous bijection theorem for saliency maps; and
2. a new characterization of the saliency map of a given hierarchy as the minimum function for which the quasi-flat zones hierarchy is precisely the given hierarchy (Th. 2); and
3. a new characterization of the minimum spanning trees of a given edge-weighted graph as the minimum subgraphs (for inclusion) whose quasi-flat zones hierarchies are the same as the one of the given graph (Th. 3).

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The links established in this article between the maps that weight the edges of a graph  $G$ , the hierarchies on the vertex set  $V(G)$  of  $G$ , the saliency maps on the edge set  $E(G)$  of  $G$ , and the minimum spanning trees for the maps that weight the edges of  $G$  are summarized in the diagram of Fig. 1.

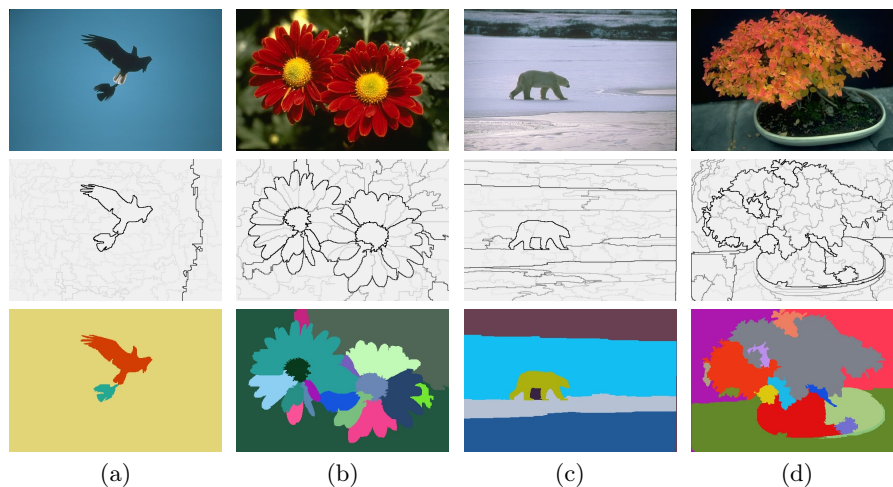


**Fig. 1.** A diagram that summarizes the results of this article. The solutions to problems  $(P_1)$ ,  $(P_2)$ , and  $(P_3)$  are given by Ths. 1, 2, and 3, respectively. The constraint involved in  $(P_2)$  and  $(P_3)$  is to leave the induced quasi-flat zones hierarchy unchanged. In the diagram,  $Q\mathcal{FZ}$  stands for quasi-flat zones (Eq. 3), and the symbols  $\Phi$  and  $\Psi$  stand for the saliency map of a hierarchy (Eq. 5) and of a map respectively (Section 5).

One possible application of these results is the design of new algorithms for computing hierarchies. Indeed, our results allow one to use indifferently any of the three hierarchical representations. This can be useful when a given operation is more efficiently performed with one representation than with the two others. Naturally, one could work directly on the hierarchy (or on its tree-based representation) and finally compute a saliency map for visualization purposes. For instance, in [8,17], the authors efficiently handle directly the tree-based representation of the hierarchy. Conversely, thanks to Th. 1, one can work on a saliency map or, thanks to Th. 3, on the weights of a minimum spanning tree and explicitly computes the hierarchy in the end. In [5,15], a resulting saliency map is computed before a possible extraction of the associated hierarchy of watersheds. In [9], a basic transformation that consists of modifying one weight on a minimum spanning tree according to some criterion is considered. The corresponding operation on the equivalent dendrogram is more difficult to design. When this basic operation is iterated on every edge of the minimum spanning tree, one transforms a given hierarchy into another one. An application of this technique to the observation scale of [7] has been developed in [9] (see Fig. 2).

Another interest of our work is to precise the link between hierarchical classification [16] and hierarchical image segmentation. In particular, it suggests that hierarchical image segmentation methods can be used for classification (the converse being carried out for a long time). Indeed, our work is deeply related to hierarchical classification, more precisely, to ultrametric distances, subdominant ultrametrics and single linkage clusterings. In classification, representation

of hierarchies, on which no connectivity hypothesis is made, are studied since the 60's (see references in [16]). The framework presented in this article deals with connected hierarchies and a graph needs to be specified for defining the connectivity of the regions of the partitions in the hierarchies. The connectivity of regions is the main difference between what has been done in classification and in segmentation. Rather than restricting the work done for classification, the framework studied in this article generalizes it. Indeed the usual notions of classification are recovered from the definitions of this article when a complete graph (every two points are linked by an edge) is considered. For instance, when a complete graph is considered, a saliency map becomes an ultrametric distance, which is known to be equivalent to a hierarchy. However, Th. 1 shows that, when the graph is not complete, we do not need a value for each pair of elements in order to characterize a hierarchy (as done with an ultrametric distance) but one value for each edge of the graph is enough (with a saliency map). Furthermore, when a complete graph is considered, the hierarchy of quasi-flat zones becomes the one of single linkage clustering. Hence, Th. 3 allows to recover and to generalize a well-known relation between the minimum spanning trees of the complete graph and single linkage clusterings.

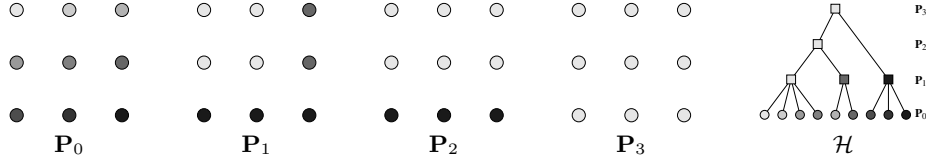


**Fig. 2.** Top row: some images from the Berkeley database [1]. Middle row: saliency maps according to [9] developed thanks to the framework of this article. Bottom row: segmentations extracted from the hierarchies with (a) 3, (b) 18, (c) 6 and (d) 16 regions.

## 2 Connected hierarchies of partitions

A *partition* of a finite set  $V$  is a set  $\mathbf{P}$  of nonempty disjoint subsets of  $V$  whose union is  $V$  (*i.e.*,  $\forall X, Y \in \mathbf{P}, X \cap Y = \emptyset$  if  $X \neq Y$  and  $\cup\{X \in \mathbf{P}\} = V$ ).

Any element of a partition  $\mathbf{P}$  of  $V$  is called a *region* of  $\mathbf{P}$ . If  $x$  is an element of  $V$ , there is a unique region of  $\mathbf{P}$  that contains  $x$ ; this unique region is denoted by  $[\mathbf{P}]_x$ . Given two partitions  $\mathbf{P}$  and  $\mathbf{P}'$  of a set  $V$ , we say that  $\mathbf{P}'$  is a *refinement* of  $\mathbf{P}$  if any region of  $\mathbf{P}'$  is included in a region of  $\mathbf{P}$ . A *hierarchy* (on  $V$ ) is a sequence  $\mathcal{H} = (\mathbf{P}_0, \dots, \mathbf{P}_\ell)$  of indexed partitions of  $V$  such that  $\mathbf{P}_{i-1}$  is a refinement of  $\mathbf{P}_i$ , for any  $i \in \{1, \dots, \ell\}$ . If  $\mathcal{H} = (\mathbf{P}_0, \dots, \mathbf{P}_\ell)$  is a hierarchy, the integer  $\ell$  is called the *depth* of  $\mathcal{H}$ . A hierarchy  $\mathcal{H} = (\mathbf{P}_0, \dots, \mathbf{P}_\ell)$  is called complete if  $\mathbf{P}_\ell = \{V\}$  and if  $\mathbf{P}_0$  contains every singleton of  $V$  (i.e.,  $\mathbf{P}_0 = \{\{x\} \mid x \in V\}$ ). The hierarchies considered in this article are complete.



**Fig. 3.** Illustration of a hierarchy  $\mathcal{H} = (\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$ . For every partition, each region is represented by a gray level: two dots with the same gray level belong to the same region. The last subfigure represents the hierarchy  $\mathcal{H}$  as a tree, often called a dendrogram, where the inclusion relation between the regions of the successive partitions is represented by line segments.

Figure 3 graphically represents a hierarchy  $\mathcal{H} = (\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3)$  on a rectangular subset  $V$  of  $\mathbb{Z}^2$  made of 9 dots. For instance, it can be seen that  $\mathbf{P}_1$  is a refinement of  $\mathbf{P}_2$  since any region of  $\mathbf{P}_1$  is included in a region of  $\mathbf{P}_2$ . It can also be seen that the hierarchy is complete since  $\mathbf{P}_0$  is made of singletons and  $\mathbf{P}_3$  is made of a single region that contains all elements.

In this article, we consider connected regions, the connectivity being given by a graph. Therefore, we remind basic graph definitions before introducing connected partitions and hierarchies.

A (*undirected*) *graph* is a pair  $G = (V, E)$ , where  $V$  is a finite set and  $E$  is composed of unordered pairs of distinct elements in  $V$ , i.e.,  $E$  is a subset of  $\{\{x, y\} \subseteq V \mid x \neq y\}$ . Each element of  $V$  is called a *vertex* or a *point* (of  $G$ ), and each element of  $E$  is called an *edge* (of  $G$ ). A *subgraph* of  $G$  is a graph  $G' = (V', E')$  such that  $V'$  is a subset of  $V$ , and  $E'$  is a subset of  $E$ . If  $G'$  is a subgraph of  $G$ , we write  $G' \subseteq G$ . The vertex and edge sets of a graph  $X$  are denoted by  $V(X)$  and  $E(X)$  respectively.

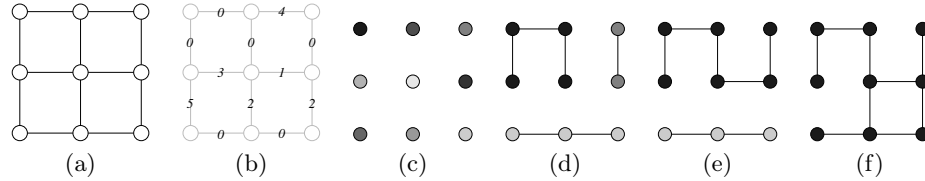
Let  $G$  be a graph and let  $(x_0, \dots, x_k)$  be a sequence of vertices of  $G$ . The sequence  $(x_0, \dots, x_k)$  is a *path* (in  $G$ ) from  $x_0$  to  $x_k$  if, for any  $i$  in  $\{1, \dots, k\}$ ,  $\{x_{i-1}, x_i\}$  is an edge of  $G$ . The graph  $G$  is *connected* if, for any two vertices  $x$  and  $y$  of  $G$ , there exists a path from  $x$  to  $y$ . Let  $X$  be a subset of  $V(G)$ . The *graph induced by  $X$*  (in  $G$ ) is the graph whose vertex set is  $X$  and whose edge set contains any edge of  $G$  which is made of two elements in  $X$ . If the graph induced by  $X$  is connected, we also say, for simplicity, that  $X$  is *connected* (for  $G$ ). The

subset  $X$  of  $V(G)$  is a *connected component* of  $G$  if it is connected for  $G$  and maximal for this property, *i.e.*, for any subset  $Y$  of  $V(G)$ , if  $Y$  is a connected superset of  $X$ , then we have  $Y = X$ . In the following, we denote by  $\mathbf{C}(G)$  the set of all connected components of  $G$ . It is well-known that this set  $\mathbf{C}(G)$  of all connected components of  $G$  is a partition of  $V(G)$ . This partition is called the *(connected components) partition induced by  $G$* . Thus, the set  $[\mathbf{C}(G)]_x$  is the unique connected component of  $G$  that contains  $x$ .

Given a graph  $G = (V, E)$ , a *partition of  $V$  is connected (for  $G$ )* if any of its regions is connected and a *hierarchy on  $V$  is connected (for  $G$ )* if any of its partitions is connected.

For instance, the partitions presented in Fig. 3 are connected for the graph given in Fig. 4(a). Therefore, the hierarchy  $\mathcal{H}$  made of these partitions, which is depicted as a dendrogram in Fig. 3 (rightmost subfigure), is also connected for the graph of Fig. 4(a).

For image analysis applications, the graph  $G$  can be obtained as a pixel or a region adjacency graph: the vertex set of  $G$  is either the domain of the image to be processed or the set of regions of an initial partition of the image domain. In the latter case, the regions can in particular be “superpixels”. In both cases, two typical settings for the edge set of  $G$  can be considered: (1) the edges of  $G$  are obtained from an adjacency relation between the image pixels, such as the well known 4- or 8-adjacency relations; and (2) the edges of  $G$  are obtained by considering, for each vertex  $x$  of  $G$ , the nearest neighbors of  $x$  for a distance in a (continuous) features space onto which the vertices of  $G$  are mapped.



**Fig. 4.** Illustration of quasi-flat zones hierarchy. (a) A graph  $G$ ; (b) a map  $w$  (numbers in black) that weights the edges of  $G$  (in gray); (c, d, e, f) the  $\lambda$ -subgraphs of  $G$ , with  $\lambda = 0, 1, 2, 3$ . The associated connected component partitions that form the hierarchy of quasi-flat zones of  $G$  for  $w$  is depicted in Fig. 3.

### 3 Hierarchies of quasi-flat zones

As established in the next sections, a connected hierarchy can be equivalently treated by means of an edge-weighted graph. We first recall in this section that the level-sets of any edge-weighted graph induce a hierarchy of quasi-flat zones. This hierarchy is widely used in image processing [12].

Let  $G$  be a graph, if  $w$  is a map from the edge set of  $G$  to the set  $\mathbb{R}^+$  of positive real numbers, then the pair  $(G, w)$  is called an *(edge-)weighted graph*. If  $(G, w)$  is an edge-weighted graph, for any edge  $u$  of  $G$ , the value  $w(u)$  is called the *weight of  $u$  (for  $w$ )*.

**Important notation.** In the sequel of this paper, we consider a weighted graph  $(G, w)$ . To shorten the notations, the vertex and edge sets of  $G$  are denoted by  $V$  and  $E$  respectively instead of  $V(G)$  and  $E(G)$ . Furthermore, we assume that the vertex set of  $G$  is connected. Without loss of generality, we also assume that the range of  $w$  is the set  $\mathbb{E}$  of all integers from 0 to  $|E| - 1$  (otherwise, one could always consider an increasing one-to-one mapping from the set  $\{w(u) \mid u \in E\}$  into  $\mathbb{E}$ ). We also denote by  $\mathbb{E}^\bullet$  the set  $\mathbb{E} \cup \{|E|\}$ .

Let  $X$  be a subgraph of  $G$  and let  $\lambda$  be an integer in  $\mathbb{E}^\bullet$ . The  $\lambda$ -level set of  $X$  (for  $w$ ) is the set  $w_\lambda(X)$  of all edges of  $X$  whose weight is less than  $\lambda$ :

$$w_\lambda(X) = \{u \in E(X) \mid w(u) < \lambda\}. \quad (1)$$

The  $\lambda$ -level graph of  $X$  is the subgraph  $w_\lambda^V(X)$  of  $X$  whose edge set is the  $\lambda$ -level set of  $X$  and whose vertex set is the one of  $X$ :

$$w_\lambda^V(X) = (V(X), w_\lambda(X)). \quad (2)$$

The connected component partition  $\mathbf{C}(w_\lambda^V(X))$  induced by the  $\lambda$ -level graph of  $X$  is called the  $\lambda$ -level partition of  $X$  (for  $w$ ).

For instance, let us consider the graph  $G$  depicted in Fig. 4(a) and the map  $w$  shown in Fig. 4(b). The 0-, 1-, 2- and 3-level sets of  $G$  contain the edges depicted in Figs. 4(c), (d), (e), and (f), respectively. The graphs depicted in these figures are the associated 0-, 1-, 2- and 3-level graphs of  $G$  and the associated 0-, 1-, 2- and 3-level partitions are shown in Fig. 3.

Let  $X$  be a subgraph of  $G$ . If  $\lambda_1$  and  $\lambda_2$  are two elements in  $\mathbb{E}^\bullet$  such that  $\lambda_1 \leq \lambda_2$ , it can be seen that any edge of the  $\lambda_1$ -level graph of  $X$  is also an edge of the  $\lambda_2$ -level graph of  $X$ . Thus, if two points are connected for the  $\lambda_1$ -level graph of  $X$ , then they are also connected for the  $\lambda_2$ -level graph of  $X$ . Therefore, any connected component of the  $\lambda_1$ -level graph of  $X$  is included in a connected component of the  $\lambda_2$ -level graph of  $X$ . In other words, the  $\lambda_1$ -level partition of  $X$  is a refinement of the  $\lambda_2$ -level partition of  $X$ . Hence, the sequence

$$\mathcal{QFZ}(X, w) = (\mathbf{C}(w_\lambda^V(X)) \mid \lambda \in \mathbb{E}^\bullet) \quad (3)$$

of all  $\lambda$ -level partitions of  $X$  is a hierarchy. This hierarchy  $\mathcal{QFZ}(X, w)$  is called the *quasi-flat zones hierarchy of  $X$  (for  $w$ )*. It can be seen that this hierarchy is complete whenever  $X$  is connected.

For instance, the quasi-flat zones hierarchy of the graph  $G$  (Fig. 4(a)) for the map  $w$  (Fig. 4(b)) is the hierarchy of Fig. 3.

For image analysis applications, we often consider that the weight of an edge  $u = \{x, y\}$  represents the dissimilarity of  $x$  and  $y$ . For instance, in the case where the vertices of  $G$  are the pixels of a grayscale image, the weight  $w(u)$  can be the absolute difference of intensity between  $x$  and  $y$ . The setting of the graph  $(G, w)$  depends on the application context.

## 4 Correspondence between hierarchies and saliency maps

In the previous section, we have seen that any edge-weighted graph induces a connected hierarchy of partitions (called the quasi-flat zones hierarchy). In this section, we tackle the inverse problem:

- ( $P_1$ ) given a connected hierarchy  $\mathcal{H}$ , find a map  $w$  from  $E$  to  $\mathbb{E}$  such that the quasi-flat zones hierarchy for  $w$  is precisely  $\mathcal{H}$ .

We start this section by defining the saliency map of  $\mathcal{H}$ . Then, we provide a one-to-one correspondence (also known as a bijection) between saliency maps and hierarchies. This correspondence is given by the hierarchy of quasi-flat zones. Finally, we deduce that the saliency map of  $\mathcal{H}$  is a solution to problem ( $P_1$ ).

Until now, we handled the regions of a partition. Let us now study their “dual” that represents “borders” between regions and that are called graph-cuts or simply cuts. The notion of a cut will then be used to define the saliency maps.

Let  $\mathbf{P}$  be a partition of  $V$ , the *cut of  $\mathbf{P}$  (for  $G$ )*, denoted by  $\phi(\mathbf{P})$ , is the set of edges of  $G$  whose two vertices belong to different regions of  $\mathbf{P}$ :

$$\phi(\mathbf{P}) = \{\{x, y\} \in E \mid [\mathbf{P}]_x \neq [\mathbf{P}]_y\}. \quad (4)$$

Let  $\mathcal{H} = (\mathbf{P}_0, \dots, \mathbf{P}_\ell)$  be a hierarchy on  $V$ . The *saliency map of  $\mathcal{H}$*  is the map  $\Phi(\mathcal{H})$  from  $E$  to  $\{0, \dots, \ell\}$  such that the weight of  $u$  for  $\Phi(\mathcal{H})$  is the maximum value  $\lambda$  such that  $u$  belongs to the cut of  $\mathbf{P}_\lambda$ :

$$\Phi(\mathcal{H})(u) = \max \{\lambda \in \{0, \dots, \ell\} \mid u \in \phi(\mathbf{P}_\lambda)\}. \quad (5)$$

In fact, the weight of the edge  $u = \{x, y\}$  for  $\Phi(\mathcal{H})$  is directly related to the lowest index of a partition in the hierarchy  $\mathcal{H}$  for which  $x$  and  $y$  belong to the same region:

$$\Phi(\mathcal{H})(u) = \min \{\lambda \in \{1, \dots, \ell\} \mid [\mathbf{P}_\lambda]_x = [\mathbf{P}_\lambda]_y\} - 1. \quad (6)$$

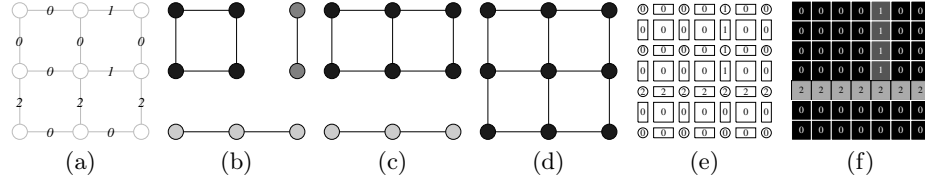
For instance, if we consider the graph  $G$  represented by the gray dots and line segments in Fig. 5(a), the saliency map of the hierarchy  $\mathcal{H}$  shown in Fig. 3 is the map shown with black numbers in Fig. 5(a). When the 4-adjacency relation is used, a saliency map can be displayed as an image (Figs. 5(e,f) and Fig. 2) which is useful for visualizing the associated hierarchy at a glance.

We say that a map  $w$  from  $E$  to  $\mathbb{E}$  is a *saliency map* if there exists a hierarchy  $\mathcal{H}$  such that  $w$  is the saliency map of  $\mathcal{H}$  (i.e.  $w = \Phi(\mathcal{H})$ ).

If  $\varphi$  is a map from a set  $S_1$  to a set  $S_2$  and if  $\varphi^{-1}$  is a map from  $S_2$  to  $S_1$  such that the composition of  $\varphi^{-1}$  with  $\varphi$  is the identity, then we say that  $\varphi^{-1}$  is the inverse of  $\varphi$ .

The next theorem identifies the inverse of the map  $\Phi$  and asserts that there is a bijection between the saliency maps and the connected hierarchies on  $V$ .

**Theorem 1** *The map  $\Phi$  is a one-to-one correspondence between the connected hierarchies on  $V$  of depth  $|E|$  and the saliency maps (of range  $\mathbb{E}$ ). The inverse  $\Phi^{-1}$  of  $\Phi$  associates to any saliency map  $w$  its quasi-flat zones hierarchy:  $\Phi^{-1}(w) = \mathcal{QFZ}(G, w)$ .*



**Fig. 5.** Illustration of a saliency map. The map (depicted by black numbers) is the saliency map  $s = \Phi(\mathcal{H})$  of the hierarchy  $\mathcal{H}$  shown in Fig. 3 when we consider the graph  $G$  depicted in gray. (b, c, d) the 1-, 2-, and 3-level graphs of  $G$  for  $s$ . The vertices are colored according to the associated 1-, 2-, and 3-level partitions of  $G$ : in each subfigure, two vertices belonging to a same connected components have the same grey level. Subfigures (e) and (f) show possible image representations of a saliency map when one considers the 4-adjacency graph.

Hence, as a consequence of this theorem, we have  $\mathcal{QFZ}(G, \Phi(\mathcal{H})) = \mathcal{H}$ , which means that  $\mathcal{H}$  is precisely the hierarchy of quasi-flat zones of  $G$  for its saliency map  $\Phi(\mathcal{H})$ . In other words, the saliency map of  $\mathcal{H}$  is a solution to problem  $(P_1)$ . For instance, if we consider the hierarchy  $\mathcal{H}$  shown in Fig. 3, it can be observed that the quasi-flat zones hierarchy for  $\Phi(\mathcal{H})$  (see Fig. 5) is indeed  $\mathcal{H}$ . We also deduce that, for any saliency map  $w$ , the relation  $\Phi(\mathcal{QFZ}(G, w)) = w$  holds true. In other words, a given saliency map  $w$  is precisely the saliency map of its quasi-flat zones hierarchy.

From this last relation, we can deduce that there are some maps that weight the edges of  $G$  and that are not saliency maps. Indeed, in general, a map is not equal to the saliency map of its quasi-flat zones hierarchy. For instance, the map  $w$  in Fig. 4 is not equal to the saliency map of its quasi-flat zones hierarchy which is depicted in Fig. 5. Thus, the map  $w$  is not a saliency map. The next section studies a characterization of saliency maps.

## 5 Characterization of saliency maps

Following the conclusion of the previous section, we now consider the problem:

- $(P_2)$  given a hierarchy  $\mathcal{H}$ , find the minimal map  $w$  such that the quasi-flat zones hierarchy for  $w$  is precisely  $\mathcal{H}$ .

The next theorem establishes that the saliency map of  $\mathcal{H}$  is the unique solution to problem  $(P_2)$ .

Before stating Th. 2, let us recall that, given two maps  $w$  and  $w'$  from  $E$  to  $\mathbb{E}$ , the map  $w'$  is less than  $w$  if we have  $w'(u) \leq w(u)$  for any  $u \in E$ .

**Theorem 2** *Let  $\mathcal{H}$  be a hierarchy and let  $w$  be a map from  $E$  to  $\mathbb{E}$ . The map  $w$  is the saliency map of  $\mathcal{H}$  if and only if the two following statements hold true:*

1. *the quasi-flat zones hierarchies for  $w$  is  $\mathcal{H}$ ; and*



2. the map  $w$  is minimal for statement 1, i.e., for any map  $w'$  such that  $w' \leq w$ , if the quasi-flat zones hierarchy for  $w'$  is  $\mathcal{H}$ , then we have  $w = w'$ .

Given a weighted graph  $(G, w)$ , it is sometimes interesting to consider the saliency map of its quasi-flat zones hierarchy. This saliency map is simply called the *saliency map of  $w$*  and is denoted by  $\Psi(w)$ . From Th. 2, the operator  $\Psi$  which associates to any map  $w$  the saliency map  $\Phi(\mathcal{QFZ}(G, w))$  of its quasi-flat zones hierarchy, is idempotent (i.e.  $\Psi(\Psi(w)) = \Psi(w)$ ). Furthermore, it is easy to see that  $\Psi$  is also anti-extensive (we have  $\Psi(w) \leq w$ ) and increasing (for any two maps  $w$  and  $w'$ , if  $w \geq w'$ , then we have  $\Psi(w) \geq \Psi(w')$ ). Thus,  $\Psi$  is a morphological opening. This operator is studied, in different frameworks, under several names (see [11,13,10,18,16]). When the considered graph  $G$  is complete, it is known in classification (see, e.g., [16]) that this operator is linked to the minimum spanning tree of  $(G, w)$ . The next section proposes a generalization of this link.

## 6 Minimum spanning trees

Two distinct maps that weight the edges of the same graph (see, e.g., the maps of Figs. 4(b) and 5(a)) can induce the same hierarchy of quasi-flat zones. Therefore, in this case, one can guess that some of the edge weights do not convey any useful information with respect to the associated quasi-flat zones hierarchy. More generally, in order to represent a hierarchy by a simple (i.e., easy to handle) edge-weighted graph with a low level of redundancy, it is interesting to consider the following problem:

- ( $P_3$ ) given an edge-weighted graph  $(G, w)$ , find a minimal subgraph  $X \subseteq G$  such that the quasi-flat zones hierarchies of  $G$  and of  $X$  are the same.

The main result of this section, namely Th. 3, provides the set of all solutions to problem ( $P_3$ ): the minimum spanning trees of  $(G, w)$ . The minimum spanning tree problem is one of the most typical and well-known problems of combinatorial optimization (see [3]) and Th. 3 provides, as far as we know, a new characterization of minimum spanning trees based on the quasi-flat zones hierarchies as used in image processing.

Let  $X$  be a subgraph of  $G$ . The weight of  $X$  with respect to  $w$ , denoted by  $w(X)$ , is the sum of the weights of all the edges in  $E(X)$ :  $w(X) = \sum_{u \in E(X)} w(u)$ . The subgraph  $X$  is a *minimum spanning tree (MST)* of  $(G, w)$  if:

1.  $X$  is connected; and
2.  $V(X) = V$ ; and
3. the weight of  $X$  is less than or equal to the weight of any graph  $Y$  satisfying (1) and (2) (i.e.,  $Y$  is a connected subgraph of  $G$  whose vertex set is  $V$ ).

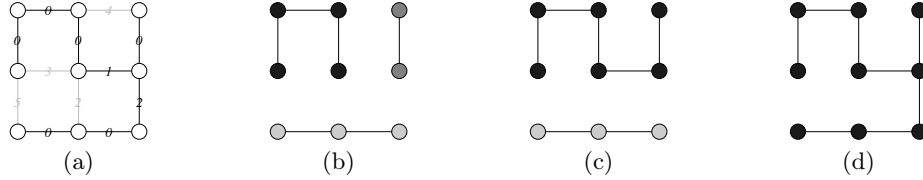
For instance, a MST of the graph shown in Fig. 4(b) is presented in Fig. 6(a).

**Theorem 3** *A subgraph  $X$  of  $G$  is a MST of  $(G, w)$  if and only if the two following statements hold true:*

1. the quasi-flat zones hierarchies of  $X$  and of  $G$  are the same; and
2. the graph  $X$  is minimal for statement 1, i.e., for any subgraph  $Y$  of  $X$ , if the quasi-flat zones hierarchy of  $Y$  for  $w$  is the one of  $G$  for  $w$ , then we have  $Y = X$ .

Theorem 3 (statement 1) indicates that the quasi-flat zones hierarchy of a graph and of its MSTs are identical. Note that statement 1 appeared in [6] but Th. 3 completes the result of [6]. Indeed, Th. 3 indicates that there is no proper subgraph of a MST that induces the same quasi-flat zones hierarchy as the initial weighted graph. Thus, a MST of the initial graph is a solution to problem  $(P_3)$ , providing a minimal graph representation of the quasi-flat zones hierarchy of  $(G, w)$ , or more generally by Th. 1 of any connected hierarchy. More remarkably, the converse is also true: a minimal representation of a hierarchy in the sense of  $(P_3)$  is necessarily a MST of the original graph. To the best of our knowledge, this result has not been stated before.

For instance, the level sets, level graphs and level partitions of the MST  $X$  (Fig. 6(a)) of the weighted graph  $(G, w)$  (Fig. 4) are depicted in Figs. 6(b), (c), (d). It can be observed that the level partitions of  $X$  are indeed the same as those of  $G$ . Thus the quasi-flat zones hierarchies of  $X$  and  $G$  are the same.



**Fig. 6.** Illustration of a minimum spanning tree and of its quasi-flat zones hierarchy. (a) A minimum spanning tree  $X$  (black edges and black circled vertices) of the weighted graph of Fig. 4(b); (b, c, d) the 1-, 2-, and 3-level graphs of  $X$ . The vertices are colored according to the associated 1-, 2-, and 3-level partitions of  $X$ : in each subfigure, two vertices belonging to the same connected components have the same color.

## 7 Saliency map algorithms

When a hierarchy  $\mathcal{H}$  is stored as a tree data structure (such as *e.g.* the dendrogram of Fig. 3), the weight of any edge for the saliency map of this hierarchy can be computed in constant time, provided a linear time preprocessing. Indeed, the weight of an edge linking  $x$  and  $y$  is associated (see Eq. 6) to the lowest index of a partition for which  $x$  and  $y$  belongs to the same region. This index can be obtained by finding the index of the least common ancestor of  $\{x\}$  and  $\{y\}$  in the tree. The algorithm proposed in [2] performs this task in constant time, provided a linear time preprocessing of the tree. Therefore, computing the saliency map of  $\mathcal{H}$  can be done in linear  $O(|E| + |V|)$  time complexity.

Thus, the complete process that computes the saliency map  $\Psi(w)$  of a given map  $w$  proceeds in two steps:

- i) build the quasi-flat zones hierarchy  $\mathcal{H} = \mathcal{QFZ}(G, w)$  of  $G$  for  $w$ ; and
- ii) compute the saliency map  $\Psi(w) = \Phi(\mathcal{H})$ .

On the basis of [6], step i) can be performed with the quasi-linear time algorithm shown in [15] and step ii) can be performed in linear-time as proposed in the previous paragraph. Thus, the overall time complexity of this algorithm is quasi-linear with respect to the size  $|E| + |V|$  of the graph  $G$ .

The algorithm sketched in [13], based on [4], for computing the saliency map of a given map  $w$  has the same complexity as the algorithm proposed above. However, the algorithm of [13] is more complicated since it requires to compute the topological watershed of the map. This involves a component tree (a data structure which is more complicated than the quasi-flat zones hierarchy in the sense of [6]), a structure for computing least common ancestors, and a hierarchical queue [4], which is not needed by the above algorithm. Hence, as far as we know, the algorithm presented in this section is the simplest algorithm for computing a saliency map. It is also the most efficient both from memory and execution-time points of view. An implementation in C of this algorithm is available at <http://www.esiee.fr/~{}info/sm>.

## 8 Conclusions

In this article we study three representations for a hierarchy of partitions. We show a new bijection between hierarchies and saliency maps and we characterize the saliency map of a hierarchy and the minimum spanning trees of a graph as minimal elements preserving quasi-flat zones. In practice, these results allow us to indifferently handle a hierarchy by a dendrogram (the direct tree structure given by the hierarchy), by a saliency map, or by an edge-weighted tree. These representations form a toolkit for the design of hierarchical (segmentation) methods where one can choose the most convenient representation or the one that leads to the most efficient implementation for a given particular operation. The results of this paper were used in [9] to provide a framework for hierarchicalizing a certain class of non-hierarchical methods. We study in particular a hierarchicalization of [7]. The first results are encouraging and a short term perspective is the precise practical evaluation of the gain of the hierarchical method with respect to its non-hierarchical counterpart.

Another important aspect of the present work is to underline and to precise the close link that exists between classification and hierarchical image segmentation. Whereas classification methods were used as image segmentation tools for a long time, our results incite us to look if some hierarchies initially designed for image segmentation can improve the processing of non-image data such as data coming from geography, social network, etc.. This topic will be a subject of future research.

## References

1. Arbelaez, P., Maire, M., Fowlkes, C., Malik, J.: Contour detection and hierarchical image segmentation. *PAMI* 33(5), 898–916 (2011)
2. Bender, M., Farach-Colton, M.: The LCA problem revisited. In: *Latin American Theoretical INformatics*. pp. 88–94 (2000)
3. Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C., et al.: *Introduction to algorithms*, vol. 2. MIT press Cambridge (2001)
4. Couprie, M., Najman, L., Bertrand, G.: Quasi-linear algorithms for the topological watershed. *JMIV* 22(2-3), 231–249 (2005)
5. Cousty, J., Najman, L.: Incremental algorithm for hierarchical minimum spanning forests and saliency of watershed cuts. In: *ISMM. LNCS*, vol. 6671, pp. 272–283 (2011)
6. Cousty, J., Najman, L., Perret, B.: Constructive links between some morphological hierarchies on edge-weighted graphs. In: *ISMM*, pp. 86–97. Springer (2013)
7. Felzenszwalb, P., Huttenlocher, D.: Efficient graph-based image segmentation. *IJCV* 59, 167–181 (2004)
8. Guigues, L., Cocquerez, J.P., Men, H.L.: Scale-sets image analysis. *IJCV* 68(3), 289–317 (2006)
9. Guimarães, S.J.F., Cousty, J., Kenmochi, Y., Najman, L.: A hierarchical image segmentation algorithm based on an observation scale. In: *SSPR/SPR*, pp. 116–125. Springer (2012)
10. Kiran, B., Serra, J.: Scale space operators on hierarchies of segmentations. In: *Scale Space and Variational Methods in Computer Vision*, vol. 7893, pp. 331–342 (2013)
11. Leclerc, B.: Description combinatoire des ultramétries. *Mathématiques et Sciences humaines* 73, 5–37 (1981)
12. Meyer, F., Maragos, P.: Morphological scale-space representation with levelings. In: *Scale-Space Theories in Computer Vision, LNCS*, vol. 1682, pp. 187–198 (1999)
13. Najman, L.: On the equivalence between hierarchical segmentations and ultrametric watersheds. *JMIV* 40(3), 231–247 (2011)
14. Najman, L., Cousty, J.: A graph-based mathematical morphology reader. *PRL* 47(1), 3–17 (2014)
15. Najman, L., Cousty, J., Perret, B.: Playing with kruskal: algorithms for morphological trees in edge-weighted graphs. In: *ISMM*, pp. 135–146. Springer (2013)
16. Nakache, J.P., Confais, J.: *Approche pragmatique de la classification: arbres hiérarchiques, partitionnements*. Editions Technip (2004)
17. Ravi Kiran, B., Serra, J.: Global–local optimizations by hierarchical cuts and climbing energies. *PR* 47(1), 12–24 (2014)
18. Ronse, C.: Ordering partial partitions for image segmentation and filtering: Merging, creating and inflating blocks. *JMIV* 49(1), 202–233 (2014)